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A Class of k-Quantum Nonlinear Coherent States and Some of Their Properties

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A class of k-quantum nonlinear coherent states, i.e., the k eigenstates of the powers $B^k (k \geq 3)$ of the annihilation operator $B = a\frac{1}{f(N)}$ of f-oscillators, is obtained and its completeness is investigated. An alternative method to construct them is proposed. We introduce a new kind of higher-order squeezing and sub-Poissonian distribution. The quantum statistical properties of the k states are studied. The result shows that all of the eigenstates can be generated by a linear superposition of k Roy-type nonlinear coherent states. These states may form a complete Hilbert space, and the M-th order $[M = (n + 1/2)k;\ n = 0, 1, \ldots]$ squeezing effects exist in all of the k states when k is even. There is the sub-Poissonian distribution in all of the states.

KEY WORDS: operator $B = a\frac{1}{f(N)}$; eigenstate; k-quantum nonlinear coherent states; completeness; higher-order squeezing; sub-Poissonian distribution.

1. INTRODUCTION

During a few last years, there has been growing interest in the nonlinear coherent states (NLCS) called f-coherent states (de Matos Filho and Vogel, 1996; Man'ko *et al.*, 1997; Roy and Roy, 1999; Aniello *et al.*, 2000; Sivakumar, 2000), which are eigenstates of the annihilation operator $A = af(n)$ of f-oscillators, where $f(n)$ is an operator-valued function of the boson number operator n. A class of f-coherent states can be realized physically as the stationary states of the centre-of-mass motion of a trapped ion (de Matos Filho and Vogel, 1996). The f-coherent states exhibit nonclassical features such as squeezing and self-splitting. The NLCS representation of the generalized Wigner operator and the P-representation of density matrix in the NLCS case were investigated by Fan and

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Cheng (2001, 2002). The phase properties of the NLCS were studied (Song and Zhu, 2002). The even and odd NLCS, which are two orthonormalized eigenstates of the square $(af(n))^2$ of the operator $A = af(n)$, were constructed and their non-classical effects were studied (Mancini, 1997; Sivakumar, 1998). On the basis of this work, the k orthonormalized eigenstates of the kth power $(af(n))^k (k \geq 1)$ were constructed and their some properties were discussed by Liu (1999). Recently the k-quantum nonlinear coherent states (KLNCS) has been introduced, whose generation schemes as well as various nonclassical properties such as multi-peaked number distribution, self-splitting, antibunching and squeezing, have been studied in detail in (Liu, 1999; Man'ko *et al.*, 2000; An, 2001a,b). The KNLCS has been shown to be physically realizable in the quantized vibration of the center-of-mass motion of a harmonically trapped ion which is further properly driven by two laser beams of which one is resonant and the other is detuned to the kth lower sideband (see e.g., Man'ko *et al.*, 2000; An, 2001a). Therefore, it is very significant to study some quantum statistical properties of the KNLCS. In practice, the k eigenstates of the operator $(af(n))^k$ is a kind of KNLCS, and quantum statistical properties of the states were investigated by us (Wang *et al.*, 2002).

Recently, a new kind of NLCS was constructed by Roy and Roy (2000) (referred as Roy-type NLCS hereafter). This kind of NLCS is the eigenstates of the operator $B = a\frac{1}{f(N)}$. On the basis of the work, the even and odd Roy-type NLCS was defined by us (Wang *et al.*, 2003), which are two orthonormalized eigenstates of the operator B^2. In this paper, we will construct k orthonormalized eigenstates of the high powers $B^k (k \geq 3)$ of the operator B, and discuss their properties and explore their generation in terms of Roy-type NLCS. In practice, the k eigenstates of the operator B^k is a class of KNLCS.

2. THE *k* ORTHONORMALIZED EIGENSTATES OF THE OPERATOR B^k

For convenience of reference and completeness, in this section we begin with some related results for the NLCS (Man'ko *et al.*, 1997) and the Roy-type NLCS (Roy and Roy, 2000).

We note that the generalized annihilation (creation) operator associated with NLCS is given by

$$A = af(N), \quad A^+ = f(N)a^+, \quad N = a^+a, \tag{1}$$

where a^+ and a are standard harmonic oscillator creation and annihilation operators and $f(x)$ is a reasonably well-behaved real function, called the nonlinearity function. From the relations Equation (1), it follows that A, A^+ and N satisfy the following closed algebraic relations:

$$[N, A] = -A, \quad [N, A^+] = A^+, [A, A^+] = f^2(N)(N+1) - f^2(N-1)N. \tag{2}$$

Clearly, the nature of the nonlinear algebra depends on the choice of the nonlinearity function $f(N)$. For $f(N) = 1$ we regain the Heisenberg algebra. NLCS $|\alpha, f\rangle$ are then defined as right eigenstates of the generalized annihilation operator A (de Matos Filho and Vogel, 1996; Man'ko *et al.*, 1997):

$$A|\alpha, f\rangle = \alpha|\alpha, f\rangle. \tag{3}$$

In the number state basis, $|\alpha, f\rangle$ is given by

$$|\alpha, f\rangle = C \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!} f(n)!}|n\rangle, \quad C = \left\{ \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n![f(n)!]^2} \right\}^{-1/2}, \tag{4}$$

where α is a complex number, and

$$f(n)! \equiv (f, n)! = f(n)f(n-1)\cdots f(1)f(0), \quad f(0) = 1. \tag{5}$$

The canonical conjugate of the generalized annihilation and creation operators A and A^+ are given by (Roy and Roy, 2000)

$$B = a\frac{1}{f(N)}, \quad B^+ = \frac{1}{f(N)}a^+. \tag{6}$$

Thus A, B^+ and their conjugates satisfy the following algebras (Roy and Roy, 2000):

$$[A, B^+] = 1, \quad [B, A^+] = 1. \tag{7}$$

In the number state basis, the Roy-type NLCS (Roy and Roy, 2000) are defined as the right eigenstates of the new generalized annihilation operator B,

$$|\beta, f\rangle = N_f \sum_{n=0}^{\infty} \frac{\beta^n f(n)!}{\sqrt{n!}}|n\rangle, \quad N_f = \left\{ \sum_{n=0}^{\infty} \frac{|\beta|^{2n}[f(n)!]^2}{n!} \right\}^{-1/2}, \tag{8}$$

where β is an arbitrary complex number.

Now, let us consider the following states:

$$|\psi_j(\beta, f)\rangle_k = C_j \sum_{n=0}^{\infty} \frac{\beta^{kn+j} f(kn+j)!}{\sqrt{(kn+j)!}}|kn+j\rangle, \tag{9}$$

where k is a positive integer (here and henceforth $k \geq 3$; we do not indicate it in the following); $j = 0, 1, 2, \ldots, k-1 \cdot C_j$ are normalized factors. With B^k operating on $|\psi_j(\beta, f)\rangle_k$, we obtain

$$B^k|\psi_j(\beta, f)\rangle_k = \beta^k C_j \sum_{n=0}^{\infty} \frac{\beta^{kn+j} f(kn+j)!}{\sqrt{(kn+j)!}}|kn+j\rangle = \beta^k|\psi_j(\beta, f)\rangle_k. \tag{10}$$

As a result, the k states of (9) are all the eigenstates of the operator B^k with same eigenvalue β^k. It is easy to check that, for the same value of k, these states are orthogonal to each other with respect to the subscript j:

$$_k\langle\psi_i(\beta, f)|\psi_j(\beta', f)\rangle_k = 0, \quad (i, j = 0, 1, 2, \ldots, k-1,\ i \neq j). \tag{11}$$

Let $x = |\beta|^2$. We easily suppose C_j to be real number. Using the normalization conditions of the states given by (9), we have

$$C_j = [B_j(x, f)]^{-1/2} = \left[\sum_{n=0}^{\infty} \frac{x^{kn+j}[f(kn+j)!]^2}{(kn+j)!}\right]^{-1/2}. \tag{12}$$

From (12) it follows that

$$\sum_{j=0}^{k-1} B_j(x, f) = \sum_{n=0}^{\infty} \frac{x^n[f(n)!]^2}{n!} = N_f^{-2} \equiv e_f(x). \tag{13}$$

3. MATHEMATICAL PROPERTIES OF THE EIGENSTATES OF THE OPERATOR B^k

Firstly, it is seen that the k eigenstates of the operator B^k contain the complex parameter β. When β takes different values, the internal product of every eigenstate does not equal zero, i.e.,

$$\begin{aligned} _k\langle\psi_j(\beta, f)|\psi_j(\beta', f)\rangle_k &= [B_j(|\beta|^2, f)B_j(|\beta'|^2, f)]^{-1/2} \sum_{m=0}^{\infty} \frac{(\beta^*\beta')^{kn+j}[f(kn+j)!]^2}{(kn+j)!} \\ &= [B_j(|\beta|^2, f)B_j(|\beta'|^2, f)]^{-1/2} B_j(\beta^*\beta', f) \neq 0, (\text{if}\ \beta \neq \beta'). \end{aligned} \tag{14}$$

This means that, in the β manifold, each of the k eigenstates of the operator B^k is not orthogonal by itself. This property is the same as that of the normal coherent states.

Secondly, in the space consisting of the k eigenstates of the operator B^k, each of the k eigenstates can be generated by the annihilation operator B. For example, if the operator B is used successively on $|\psi_0(\beta, f)\rangle_k$, we have

$$B^i|\psi_0(\beta, f)\rangle_k = \beta^i B_0^{-1/2}(|\beta|^2, f)B_{k-i}^{1/2}(|\beta|^2, f)|\psi_{k-i}(\beta, f)\rangle_k, (i = 1, 2, \ldots, k). \tag{15}$$

That is, under the action of B, the eigenstate $|\psi_0(\beta, f)\rangle_k$ may be transformed in turn as follows: $|\psi_0(\beta, f)\rangle_k \to |\psi_{k-1}(\beta, f)\rangle_k \to |\psi_{k-2}(\beta, f)\rangle_k \to \cdots \to |\psi_1(\beta, f)\rangle_k \to |\psi_0(\beta, f)\rangle_k$. Therefore, the operator B plays the role of a 'rotation operator' among the k eigenstates of the operator B^k.

The final question that concerns us is whether the k states given by (9) could construct a complete Hilbert space, i.e., whether they could be used as a representation. In order to construct the completeness formula of the k states, we use the density operator method (Hao, 1993). We define the density operator (i.e., a density matrix) of the state $|kn + j\rangle$:

$$\rho_j = \sum_{n=0}^{\infty} P(kn + j)|kn + j\rangle\langle kn + j|, \tag{16}$$

where $P(kn + j) = \int P(kn + j, \beta)d^2\beta$ is the probability distribution of the $(kn + j)$th state $|kn + j\rangle$ in the state $|\psi_j(\beta, f)\rangle_k$ in which

$$P(kn + j, \beta) = |\langle kn + j|\psi_j(\beta, f)\rangle_k|^2 = \frac{1}{B_j(|\beta|^2, f)} \frac{|\beta|^{2(kn+j)}[f(kn + j)!]^2}{(kn + j)!}. \tag{17}$$

Thus, we have $\rho_j^{-1} = \sum_{n=0}^{\infty} P^{-1}(kn + j)|kn + j\rangle\langle kn + j|$. Therefore, the completeness formula of the k states given by (9) can be written as

$$\sum_{j=0}^{k-1} \rho_j^{-1} \int d^2\beta|\psi_j(\beta, f)\rangle_k \cdot {}_k\langle\psi_j(\beta, f)| = 1. \tag{18}$$

The proof of the Equation (18) is given as following:

$$\begin{aligned}
&\sum_{j=0}^{k-1} \rho_j^{-1} \int d^2\beta|\psi_j(\beta, f)\rangle_k \cdot {}_k\langle\psi_j(\beta, f)| = \sum_{j=0}^{k-1} \rho_j^{-1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{f(km + j)!f(kn + j)!}{\sqrt{(km + j)!(kn + j)!}} \\
&\quad\times \int d^2\beta \frac{\beta^{km+j}\beta^{*(kn+j)}}{B_j(|\beta|^2, f)} |km + j\rangle\langle kn + j| \\
&= \sum_{j=0}^{k-1} \rho_j^{-1} \sum_{n=0}^{\infty} 2\pi \int r\, dr \frac{(r^2)^{kn+j}[f(kn + j)!]^2}{B_j(r^2, f)(kn + j)!} |kn + j\rangle\langle kn + j| \\
&= \sum_{j=0}^{k-1} \rho_j^{-1} \sum_{n=0}^{\infty} P(kn + j)|kn + j\rangle\langle kn + j| \\
&= \sum_{j=0}^{k-1}\sum_{m=0}^{\infty} P^{-1}(km + j)|km + j\rangle\langle km + j| \sum_{n=0}^{\infty} P(kn + j)|kn + j\rangle\langle kn + j| \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n| = 1,
\end{aligned} \tag{19}$$

where $\beta = r \exp(i\theta)$, $d^2\beta = r dr d\theta$. Therefore, the linear combination of the k states may form a complete representation. They can be used as a representation. For example, in this representation, the Roy-type NLCS $|\beta, f\rangle$ (Roy and Roy,

2000) [see Equation (8)] may be expressed as

$$|\beta, f\rangle = N_f^{-1/2} \sum_{j=0}^{k-1} B_j^{1/2}(|\beta|^2, f)|\psi_j(\beta, f)\rangle_k. \tag{20}$$

It is interesting to note that when $f(n) \to 1$, the k states given by (9) become k orthonormalized eigenstates of the high powers of the annihilation operator a of the usual harmonic oscillator (Sun *et al.*, 1991, 1992).

4. GENERATION OF THE EIGENSTATES OF THE OPERATOR B^k

According to (20), we consider the following k Roy-type NLCS:

$$|\beta_l, f\rangle = |\beta e^{i2\pi l/k}, f\rangle = e_f^{-1/2}(|\beta|^2) \sum_{n=0}^{\infty} \frac{\beta^n f(n)!}{\sqrt{n!}} e^{i(2\pi/k)nl}|n\rangle,$$
$$(l = 0, 1, 2, \ldots, k-1). \tag{21}$$

The k Roy-type NLCS are discretely distributed with an equal interval of angle along a circle around the origin of the β-plane. The inner product of the two states of (21) is

$$\langle \beta_l, f \,|.\, \beta_{l'}, f\rangle = e_f^{-1}(|\beta|^2) e_f\left(|\beta|^2 e^{i2\pi(l'-l)/k}\right), \quad (l, l' = 0, 1, 2, \ldots, k-1). \tag{22}$$

Consider a linear transformation S such that

$$|\varphi\rangle_k = S|\beta, f\rangle_k \tag{23}$$

where

$$|\beta, f\rangle_k = \begin{pmatrix} |\beta_0, f\rangle \\ |\beta_1, f\rangle \\ \vdots \\ |\beta_{k-1}, f\rangle \end{pmatrix}, \quad |\varphi\rangle_k = \begin{pmatrix} |\varphi_0\rangle_k \\ |\varphi_1\rangle_k \\ \vdots \\ |\varphi_{k-1}\rangle_k \end{pmatrix}. \tag{24}$$

S is a $k \times k$ matrix that makes φ_j orthonormal, and ${}_k\langle\varphi_j|\varphi_{j'}\rangle_k = \delta_{jj'}$. The above requirement leads to a set of algebraic equations for S_{ij},

$$\sum_{l=0}^{k-1} \sum_{l'=0}^{k-1} e_f^{-1}(|\beta|^2) e_f\left(|\beta|^2 e^{i(2\pi/k)(l'-l)}\right) S_{jl}^* S_{j'l'} = \delta_{jj'}. \tag{25}$$

The solution of Equation (25), S_{ij}, can be found as follows. By virtue of the relation

$$\sum_{l'=0}^{k-1} e_f\left(|\beta|^2 e^{\pm i(2\pi/k)(l'-l)}\right) e^{-i(2\pi/k)jl'} = e^{-i(2\pi/k)jl}$$

$$\sum_{l'=0}^{k-1} e_f\left(|\beta|^2 e^{\pm i(2\pi/k)l'}\right) e^{-i(2\pi/k)jl'} \tag{26}$$

the matrix elements of S that satisfy (25) are given by

$$\begin{aligned} S_{jl} &= \frac{1}{k} e_f^{1/2}(|\beta|^2) \left[\frac{1}{k} \sum_{l'=0}^{k-1} e_f\left(|\beta|^2 e^{i(2\pi/k)l'}\right) e^{-i(2\pi/k)jl} \right]^{-1/2} e^{-i(2\pi/k)jl} \\ &= \frac{1}{k} e_f^{1/2}(|\beta|^2) B_j^{-1/2}(|\beta|^2, f) e^{-i(2\pi/k)jl}, (j, l = 0, 1, 2, \ldots, k-1). \end{aligned} \tag{27}$$

From (23) and (27), we obtain k orthonormalized states

$$|\varphi_j\rangle_k = \frac{1}{k} B_j^{-1/2}(|\beta|^2, f) e_f^{1/2}(|\beta|^2) \sum_{l=0}^{k-1} e^{-i(2\pi/k)jl} \left|\beta e^{i(2\pi k)l}, f\right\rangle, \quad (j = 0, 1, \ldots, k-1). \tag{28}$$

which are just what we want. By use of the relation

$$\sum_{l=0}^{k-1} e^{i(2\pi/k)lt} = 0, \quad (t = 1, 2, \ldots, k-1), \tag{29}$$

it can be proved that

$$|\varphi_j\rangle_k = |\psi_j(\beta, f)\rangle_k, \quad (j = 0, 1, \ldots, k-1). \tag{30}$$

According to (28), for $k = 2$, we obtain

$$|\varphi_0\rangle_2 = \frac{1}{2} B_0^{-1/2}(|\beta|^2, f) e_f^{1/2}(|\beta|^2)(|\beta, f\rangle + |-\beta, f\rangle), \tag{31}$$

$$|\varphi_1\rangle_2 = \frac{1}{2} B_1^{-1/2}(|\beta|^2, f) e_f^{1/2}(|\beta|^2)(|\beta, f\rangle - |-\beta, f\rangle), \tag{32}$$

which are just the so-called even and odd Roy-type NLCS defined by us (Wang *et al.*, 2003).

The $|\varphi_j\rangle_k (j = 0, 1, \ldots, k-1)$ in (28) are exactly the k orthonormalized eigenstates of the operator B^k obtained in Section 2, but reconstructed here by a different method. From the above reconstruction, we come to an important conclusion that any orthonormalized eigenstates of the operator B^k can be generated from a linear superposition of k Roy-type NLCS $|\beta e^{i(2\pi/k)l}, f\rangle (l = 0, 1, 2, \ldots, k-1)$, which have the same amplitude but different phases. Yet, from (28), one can find the connection between the Roy-type NLCS and these k orthonormalized eigenstates of the operator B^k. In practice, the k eigenstates of the operator B^k is a new kind of KNLCS.

5. HIGHER-ORDER SQUEEZING OF THE EIGENSTATES OF THE OPERATOR B^k

5.1. Definition of Higher-Order Squeezing

In analogy to the definition of higher-order squeezing for the conventional single mode of the electromagnetic field (Zhang *et al.*, 1990), we define two Hermite operators

$$W_1(M) = (B^{+M} + B^M)/2, \quad W_2(M) = i(B^{+M} - B^M)/2. \tag{33}$$

It can be proved that the operators $W_1(M)$ and $W_2(M)$ satisfy the commutation relation

$$[W_1(M), W_2(M)] = (i/2)[B^M, B^{+M}], \tag{34}$$

and the uncertainty relation

$$\langle(\Delta W_1)^2\rangle \cdot \langle(\Delta W_2)^2\rangle \geq \frac{1}{16}|\langle[B^M, B^{+M}]\rangle|^2. \tag{35}$$

A state is squeezed to order M if

$$\langle(\Delta W_i)^2\rangle - \frac{1}{4}|\langle[B^M, B^{+M}]\rangle| < 0, \quad (i = 1, 2). \tag{36}$$

Form (33) and (36), we can see that it is the higher-order squeezing defined by Zhang *et al.* (1990) when $f(n) \to 1$. Therefore, this kind of Mth-order squeezing is a natural generalization of the higher-order squeezing defined by Zhang *et al.* It is formally similar to the higher-order squeezing defined by Zhang *et al.*

5.2. Properties of Higher-Order Squeezing of the Eigenstates of Operator B^k

Now we study the properties of the Mth-order squeezing for the k eigenstates given by (9) in four cases.

5.2.1. When $M = nk(n = 1, 2, 3, \cdots)$, For Even and Odd k

In this situation, for all of the states given by (9), we have

$$_k\langle\psi_j(\beta, f)|B^{+2M}|\psi_j(\beta, f)\rangle_k = r^{2nk}e^{-i2nk\theta}, \tag{37a}$$

$$_k\langle\psi_j(\beta, f)|B^{2M}|\psi_j(\beta, f)\rangle_k = r^{2nk}e^{i2nk\theta}, \tag{37b}$$

$$_k\langle\psi_j(\beta, f)|B^{+M}|\psi_j(\beta, f)\rangle_k = r^{nk}e^{-ink\theta}, \tag{37c}$$

$$_k\langle\psi_j(\beta, f)|B^{M}|\psi_j(\beta, f)\rangle_k = r^{nk}e^{ink\theta}, \tag{37d}$$

$$_k\langle\psi_j(\beta, f)|B^{+M}B^M|\psi_j(\beta, f)\rangle_k = r^{2nk}. \tag{37e}$$

Substituting (37a)–(37e) into (36), for the states given by (9), this reads

$$ {}_k\langle\psi_j(\beta, f)|(\Delta W_i)^2|\psi_j(\beta, f)\rangle_k - \frac{1}{4}\cdot|{}_k\langle\psi_j(\beta, f)|[B^M, B^{+M}]| \psi_j(\beta, f)\rangle_k| = 0, \quad (i = 1, 2). \tag{38} $$

This indicates that the k states of (9) are all minimum uncertainty states of the operators $W_1(M)$ and $W_2(M)(M = nk, n = 1, 2, \ldots)$ defined by (33).

5.2.2. When $M = nk + i$ $(n = 0, 1, 2, \ldots; i = 1, 2, \cdots, k-1)$, For Odd k

In these conditions, for all k states of (9), we have

$$ {}_k\langle\psi_j(\beta, f)|B^{+2M}|\psi_j(\beta, f)\rangle_k = {}_k\langle\psi_j(\beta, f)|B^{2M}|\psi_j(\beta, f)\rangle_k = {}_k\langle\psi_j(\beta, f)|B^{+M}|\psi_j(\beta, f)\rangle_k = {}_k\langle\psi_j(\beta, f)|B^{M}|\psi_j(\beta, f)\rangle_k = 0. \tag{39} $$

Using relation (15), we obtain

$$ {}_k\langle\psi_S(\beta, f)|B^{+M}B^M|\psi_S(\beta, f)\rangle_k = r^{2(nk+i)}B_{k-i+S}/B_S, \quad (S = 0, 1, 2, \ldots, i-1), \tag{40} $$

$$ {}_k\langle\psi_t(\beta, f)|B^{+M}B^M|\psi_t(\beta, f)\rangle_k = r^{2(nk+i)}B_{t-i}/B_t, \quad (t = i, i+1, \ldots, k-1). \tag{41} $$

Therefore, for the states $|\psi_S(\beta, f)\rangle_k (S = 0, 1, 2, \ldots, i-1)$ and $|\psi_t(\beta, f)\rangle_k$ $(t = i, i+1, \ldots, k-1)$, we have

$$ {}_k\langle\psi_S(\beta, f)|(\Delta W_1)^2|\psi_S(\beta, f)\rangle_k - \frac{1}{4}|{}_k\langle\psi_S(\beta, f)|[B^M, B^{+M}]|\psi_S(\beta, f)\rangle_k| = \frac{1}{2}r^{2(nk+i)}B_{k-i+S}/B_S, \tag{42} $$

$$ {}_k\langle\psi_t(\beta, f)|(\Delta W_1)^2|\psi_t(\beta, f)\rangle_k - \frac{1}{4}|{}_k\langle\psi_t(\beta, f)|[B^M, B^{+M}]|\psi_t(\beta, f)\rangle_k| = \frac{1}{2}r^{2(nk+i)}B_{t-i}/B_t, \tag{43} $$

According to (5) and f is chosen to be nonnegative, from (12) we have $B_j(r^2, f) > 0$ when $r = |\beta| \neq 0$. Then the right-hand sides of (42) and (43) are larger than zero. Therefore, none of the k eigenstates given by (9) exhibits Mth-order

($M = kn + i$; $n = 0, 1, 2, \ldots$; $i = 1, 2, \ldots, k - 1$) squeezing effect in these conditions (the results for W_2 can be obtained in the same way).

5.2.3. When $M = nk + i$ ($n = 0, 1, \ldots$; $i = 1, 2, \ldots, k/2 - 1, k/2 + 1, \ldots, k - 1$), For Even k

With the above discussion, it can be proved that under these conditions, none of the k states given by (9) has Mth-order squeezing.

5.2.4. When $M = (n + 1/2)k$ ($n = 0, 1, 2, \ldots$), For Even k

In these conditions, we have

$$_k\langle\psi_j(\beta, f)|B^{+2M}|\psi_j(\beta, f)\rangle_k = r^{(2n+1)k}e^{-i(2n+1)k\theta}, \tag{44a}$$

$$_k\langle\psi_j(\beta, f)|B^{2M}|\psi_j(\beta, f)\rangle_k = r^{(2n+1)k}e^{i(2n+1)k\theta}, \tag{44b}$$

$$_k\langle\psi_j(\beta, f)|B^{+M}|\psi_j(\beta, f)\rangle_k = {}_k\langle\psi_j(\beta, f)|B^{M}|\psi_j(\beta, f)\rangle_k = 0. \tag{44c}$$

Making use of (15), we can obtain

$$_k\langle\psi_S(\beta, f)|B^{+M}B^{M}|\psi_S(\beta, f)\rangle_k = r^{(2n+1)k}B_{k/2+S}/B_S, \quad (S = 0, 1, 2, \ldots, k/2 - 1), \tag{45}$$

$$_k\langle\psi_t(\beta, f)|B^{+M}B^{M}|\psi_t(\beta, f)\rangle_k = r^{(2n+1)k}B_{t-k/2}/B_t, \quad (t = k/2, k/2 + 1, \ldots, k - 1). \tag{46}$$

Consequently, for the state $|\psi_S(\beta, f)\rangle_k (S = 0, 1, \cdots, k/2 - 1)$ and $|\psi_t(\beta, f)\rangle_k (t = k/2,\ k/2 + 1, \ldots, k - 1)$ we have, respectively,

$$_k\langle\psi_S(\beta, f)|(\Delta W_1)^2|\psi_S(\beta, f)\rangle_k - \frac{1}{4}|_k\langle\psi_S(\beta, f)|[B^M, B^{+M}]|\psi_S(\beta, f)\rangle_k| = \frac{1}{2}r^{(2n+1)k}[B_{k/2+S}/B_S + \cos(2n + 1)k\theta], \tag{47}$$

$$_k\langle\psi_t(\beta, f)|(\Delta W_1)^2|\psi_t(\beta, f)\rangle_k - \frac{1}{4}|_k\langle\psi_t(\beta, f)|[B^M, B^{+M}]|\psi_t(\beta, f)\rangle_k| = \frac{1}{2}r^{(2n+1)k}[B_{t-k/2}/B_t + \cos(2n + 1)k\theta]. \tag{48}$$

According to (47) and (48), the conditions which ensure the existence of the Mth-order [$M = (n + 1/2)k,\ n = 0, 1, 2, \ldots$] squeezing in the states $|\psi_S(\beta, f)\rangle_k$

$(S = 0, 1, 2, \ldots, k/2 - 1)$ and $|\psi_t(\beta, f)\rangle_k$ $(t = k/2, k/2 + 1, \ldots, k - 1)$ are

$$B_{k/2+S}/B_S + \cos(2n + 1)k\theta < 0, \tag{49}$$

$$B_{t-k/2}/B_t + \cos(2n + 1)k\theta < 0. \tag{50}$$

Choose $\theta = \pi/[(2n + 1)k]$ (or $\theta = 2\pi/[(2n + 1)k]$), so that $\pm\cos[(2n + 1)k\theta] = -1$. From (12), we have $B_{k/2+S}/B_S < 1$ when $r = |\beta| \leq 1$. Thus, (49) holds for $r \leq 1$. For $k \geq 3$, in the regions of $r > 1$, there certainly exist such values of r that $B_{t-k/2}/B_t < 1$. Therefore, Equation (50) holds. In summary, there exists an Mth-order $[M = (n + 1/2)k;\ n = 0, 1, 2, \ldots]$ squeezing effect among the k eigenstates given by (9) for even k. The results for the direction W_2 can also be obtained in the same way.

6. AN ANALOGUE OF THE SUB-POISSONIAN FOR THE EIGENSTATES OF THE OPERATOR B^k

In analogy with the definition of sub-Poissonian distribution (Walls, 1983) for photon statistic properties of the radiation field, we introduce the second-order quantum-correlation function for the k states given by (9) as

$$g_j^{(2)}(0) = \frac{{}_k\langle\psi_j(\beta, f)|B^{+2}B^2|\psi_j(\beta, f)\rangle_k}{{}_k\langle\psi_j(\beta, f)|B^+B|\psi_j(\beta, f)\rangle_k^2}, \quad (j = 0, 1, 2, \ldots, k - 1). \tag{51}$$

The states $|\psi_j(\beta, f)\rangle_k$ are said to be showing sub-Poissonian distribution if $g_j^{(2)}(0) < 1$. From (51), we can see that the sub-Poissonian distribution of a light field (Walls, 1983) occurs when $f(n) \to 1$. Therefore, this kind of sub-Poissonian distribution is a natural generalization of the sub-Poissonian distribution of a light field. It is formally similar to the sub-Poissonian distribution of a light field (Walls, 1983).

Now we study the properties of the sub-Poissonian distribution for the k eigenstates given by (9).

Using (15) and (51), for the k states given by (9), we obtain

$$g_0^{(2)}(0) = \frac{{}_k\langle\psi_0(\beta, f)|B^{+2}B^2|\psi_0(\beta, f)\rangle_k}{{}_k\langle\psi_0(\beta, f)|B^+B|\psi_0(\beta, f)\rangle_k^2} = \frac{B_0 B_{k-2}}{B_{k-1}^2}, \tag{52}$$

$$g_1^{(2)}(0) = \frac{{}_k\langle\psi_1(\beta, f)|B^{+2}B^2|\psi_1(\beta, f)\rangle_k}{{}_k\langle\psi_1(\beta, f)|B^+B|\psi_1(\beta, f)\rangle_k^2} = \frac{B_1 B_{k-1}}{B_0^2}, \tag{53}$$

$$g_j^{(2)}(0) = \frac{{}_k\langle\psi_j(\beta, f)|B^{+2}B^2|\psi_j(\beta, f)\rangle_k}{{}_k\langle\psi_j(\beta, f)|B^+B|\psi_j(\beta, f)\rangle_k^2} = \frac{B_{j-2} B_j}{B_{j-1}^2}, \quad (j = 2, 3, \ldots, k - 1). \tag{54}$$

Substituting (12) into (52), it follows that

$$g_0^{(2)}(0) = \frac{\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}\frac{[f(kn)!]^2[f(km-kn+k-2)!]^2}{(kn)!(km-kn+k-2)!}\right\}x^{km}}{x^k\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}\frac{[f(kn+k-1)!]^2[f(km-kn+k-1)!]^2}{(kn+k-1)!(km-kn+k-1)!}\right\}x^{km}}$$
$$= \varphi_1(x)/[x^k\varphi_2(x)], \tag{55}$$

where $x = r^2 = |\beta|^2$. When $f(i) \geq f(i+1)$, for $k \geq 3$, we have

$$\sum_{n=0}^{m}\frac{[f(kn)!]^2[f(km-kn+k-2)!]^2}{(kn)!(km-kn+k-2)!} > \sum_{n=0}^{m}\frac{[f(kn+k-1)!]^2[f(km-kn+k-1)!]^2}{(kn+k-1)!(km-kn+k-1)!}, \tag{56}$$

and thus $\varphi_1(x) > \varphi_2(x)$ for $x > 0$ when $f(i) \geq f(i+1)$. Hence $g_0^{(2)}(0) > 1$ when $x \leq 1$. However, when $x > 1$ and $f(i) \geq f(i+1)$, there certainly exist values of x [e.g., $x^k > \varphi_1(x)/\varphi_2(x)$] for which the following relation holds

$$g_0^{(2)}(0) = \varphi_1(x)/[x^k\varphi_2(x)] < 1. \tag{57}$$

Substituting (12) into (53), we have

$$g_1^{(2)}(0) = \frac{x^k\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}\frac{[f(kn+1)!]^2[f(km-kn+k-1)!]^2}{(kn+1)!(km-kn+k-1)!}\right\}x^{km}}{\sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}\frac{[f(kn)!]^2[f(km-kn)!]^2}{(kn)!(km-kn)!}\right]x^{km}}$$
$$= x^k\varphi_3(x)/\varphi_4(x). \tag{58}$$

Obviously, when $f(i) \geq f(i+1)$ we have

$$\sum_{n=0}^{m}\frac{[f(kn+1)!]^2[f(km-kn+k-1)!]^2}{(kn+1)!(km-kn+k-1)!} < \sum_{n=0}^{m}\frac{[f(kn)!]^2[f(km-kn)!]^2}{(kn)!(km-kn)!}, \tag{59}$$

so $\varphi_3(x) < \varphi_4(x)$. Therefore, $g_1^{(2)}(0) < 1$ when $x^k < \varphi_4(x)/\varphi_3(x)$ and $f(i) \geq f(i+1)$.

From (12) and (54), we obtain

$$g_j^{(2)}(0) = \frac{\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}\frac{[f(kn+j-2)!]^2[f(km-kn+j)!]^2}{(kn+j-2)!(km-kn+j)!}\right\}x^{km}}{\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}\frac{[f(kn+j-1)!]^2[f(km-kn+j-1)!]^2}{(kn+j-1)!(km-kn+j-1)!}\right\}x^{km}}. \tag{60a}$$

When $f(i) \geq f(i+1)$, we have

$$g_j^{(2)}(0) < \frac{\sum_{m=0}^{\infty} \frac{(m+1)[f(j-2)!]^2[f(j)!]^2}{(j-2)!j!} x^{km}}{\sum_{m=0}^{\infty} \frac{(m+1)[f(km+j-1)!]^4}{[(km+j-1)!]^2}}$$

$$< \frac{\frac{[f(j)!]^2[f(j-2)!]^2}{j!(j-2)!} \sum_{m=0}^{\infty}(m+1)x^{km}}{\frac{[f(j-1)!]^4}{[(j-1)!]^2}}, (j = 2, 3, \ldots, k-1). \quad (60b)$$

Obviously,

$$\lim_{x \to 0} \sum_{m=0}^{\infty} (m+1)x^{km} = 1. \quad (61)$$

Therefore, from (60) and (61), we obtain

$$\lim_{x \to 0} g_j^{(2)}(0) < \frac{\{(j-1)![f(j)!][f(j-2)!]\}^2}{j!(j-2)![f(j-1)!]^4}$$

$$= \frac{(j-1)[f(j)]^2}{j[f(j-1)]^2}, \quad (j = 2, 3, \ldots, k-1). \quad (62)$$

It can be seen that there is the sub-Poissonian distribution in the states $|\psi_j(\beta, f)\rangle_k (j = 2, 3, \ldots, k-1)$ when $x = |\beta|^2 \to 0$ and $f(j-1) \geq f(j)$.

We sum up the above results and obtain that, in some different ranges of $x = |\beta|^2$, there is the sub-Poissonian distribution in all of the k states given by (9) when $f(i) \geq f(i+1)$.

7. CONCLUSION AND DISCUSSIONS

In this paper, the k orthonormalized eigenstates of the powers $B^k (B = a\frac{1}{f(N)}$, $k \geq 3)$ of the annihilation operator $B = a\frac{1}{f(N)}$ of f-oscillators are obtained and their completeness is investigated. An alternative method to construct them is proposed. We defined the higher-order squeezing and the sub-Poissonian distribution. The quantum statistical properties of the k states are studied. According to the above discussions, for the k eigenstates of the operator B^k, we come to the following conclusions: (i) The linear combination of them may form a complete Hilbert space. (ii) All of them can be generated by a linear superposition of k Roy-type NLCS. (iii) For odd k, none of them has the higher-order squeezing. (iv) For odd k and even k, all of them are the minimum uncertainty states of the operator $W_1(M)$ and $W_2(M)$ $(M = nk, n = 0, 1, 2, \ldots)$ defined by Equation (33). (v) For even k, when $M = (n + 1/2)k$ $(n = 0, 1, 2, \ldots)$, all of them exhibit the Mth-order squeezing effect. (vi) In some different ranges of $|\beta|^2$, there is the sub-Poissonian distribution in all of the k eigenstates when $f(i) \geq f(i+1)$.

It is well known that a class of NLCS can be realized physically as the stationary states of the center-of-mass motion of a trapped ion (de Matos Filho and Vogel, 1996). These experimental requirements could be fulfilled using available trapped ion techniques (Monroe *et al.*, 1996; Monroe *et al.*, 1995; Diedrich *et al.*, 1989). Similarly, the KNLCS has been shown to be physically realizable in the quantized vibration of the center-of-mass motion of a harmonically trapped ion which is further properly driven by two laser beams of which one is resonant and the other is detuned to the kth lower sideband (Man'ko *et al.*, 2000; An, 2001a). In fact, the k eigenstates of the operator B^k investigated by this paper are a new kind of KNLCS. This kind of KNLCS will reduce to the Roy-type NLCS defined by Equation (8) when $k = 1$. Therefore, we construct this kind of KNLCS and study some quantum statistical properties of them, which are very significant. The constructing this kind of KNLCS provides a means to manipulate the quantum state. This would give relevant motivation for more thorough studies in the future.

In addition, it is interesting to note that when $f(n) \to 1$, the k orthonormalized eigenstates of the operator B^k become the states investigated by us in Sun *et al.*, (1991, 1992).

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